Waves & Oscillations

for B.Sc. Students

Supplement to mathematical portion of the
Textbook by Resnick/Halliday/Krane

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PREFACE

This book of "Waves & Oscillations" contains mathematical details which should be beneficial to students. I want to offer some words of advice which is based upon my 26 years of experience in teaching at college level.

You should read the text carefully. It is not possible for an ordinary student to absorb the full meaning of scientific writing after one reading. Several readings of the text with class lectures are necessary.

The discussion with your class fellows will also increase your understanding about the subject. The class lecture will be very much meaningful if you read the corresponding text in advance.

I have tried to keep such students in mind that has inadequate mathematical background by writing mathematical details at some places as a footnote or side-note.
Please give your suggestions and criticisms. I will take into consideration for next edition.

August 20, 2003 Ross Nazir Ullah
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CONTENTS

HARMONIC OSCILLATIONS

1-1: DEFINITIONS:

Translatory motion: A body moves with translatory motion if each particle of the body undergoes the same displacement in a straight line in a given time.

Rotatory motion: A body moves with rotatory motion if each particle of the body moves in a circle about a straight line called the axis of rotation,

Vibratory motion: If the motion is back and forth over the same path about a mean position, it is called vibratory or oscillatory motion.

Simple Harmonic Motion: 1) The projection of uniform circular motion upon any diameter of a circle.
2) A particle is said to possess a simple harmonic motion if its acceleration is always directed towards the centre and its value is proportional to the displacement of the particle from its central position.

Periodic motion: A motion which repeats itself in equal intervals of time.

Vibration: One complete round trip of the body.

Time period: It is the time required to complete one vibration.

Frequency: It is the number of vibratiuons executed by a body in one second.

Displacement: Distance from the equilibrium position at certain instant.

Amplitude: The maximum distance travelled by a vibrating particle from its mean position.

Wavelength: The distance between one particle in a wave and the corresponding particle in the next wave.

Phase: The state or condition as regards its position and direction of motion with respect to the mean position.

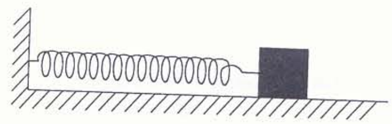
Waveform: The shape of a signal or wave displayed on a cathode-ray tube or other recording device.

Sinusoidal waveform: A waveform consisting of sine wave.

Hooke's Law: Within the limits of perfect elasticity strain is directly proportional to stress.

1-2: SIMPLE HARMONIC MOTION:

Consider a simple harmonic oscillator, consisting of a spring acting on a body that slides on a frictionless horizontal surface.



From modified form of Hooke's Law,

Restoring force: $F = -kx$ .....(1.1)

From Newton's 2nd Law,

Applied force: $F = ma$ .....(1.2)

From eqs (1.1) & (1.2), we get

$$ma = -kx$$

$$m\,(d^2x/dt^2) = -kx \qquad [\,a = d^2x/dt^2$$

$$\text{or} \quad d^2x/dt^2 + (k/m)\,x = 0 \qquad .....(1.3)$$

Relation (1.3) is the <u>equation of motion of the simple harmonic oscillator.</u>

$$\text{Put} \quad k/m = \omega^2 \qquad .....(1.4)$$

$$\text{Then} \quad d^2x/dt^2 + \omega^2 x = 0 \qquad .....(1.5)$$

$$\text{or} \quad \ddot{x} + \omega^2 x = 0$$

$$\dot{x}\ddot{x} + \omega^2 \dot{x} x = 0$$

$$\tfrac{1}{2}\, d/dt\,(\dot{x}^2) + \tfrac{1}{2}\,\omega^2\, d/dt\,(x^2) = 0$$

$$d/dt\,\{\tfrac{1}{2}\dot{x}^2 + \tfrac{1}{2}\omega^2 x^2\} = 0$$

$$\left[\;\dot{x} = \frac{dx}{dt}, \qquad \frac{d}{dt}x^2 = 2x\dot{x}\right.$$

integrating w.r.t. 't',

$$\tfrac{1}{2}\dot{x}^2 + \tfrac{1}{2}\omega^2 x^2 = B$$

$$\dot{x}^2 = 2B - \omega^2 x^2$$

$$\dot{x} = \sqrt{(2B - \omega^2 x^2)}$$

$$dx/dt = \sqrt{(2B - \omega^2 x^2)}$$

$$dx/(\sqrt{2B - \omega^2 x^2}) = dt$$

$$\int_{x_o}^{x} dx/(\sqrt{2B - \omega^2 x^2}) = \int_{o}^{t} dt \qquad \left[\int dx/\sqrt{a^2 - x^2} = \sin^{-1} x/a\right.$$

$$(1/\omega)\,[\sin^{-1}(\omega x)/\sqrt{2B}\,]_{x_o}^{x} = t$$

$$\sin^{-1}(\omega x)/\sqrt{2B} - \sin^{-1}(\omega x_o)/\sqrt{2B} = \omega t$$

$$\text{put} \quad \sin^{-1}(\omega x_o)/\sqrt{2B} = \phi_1$$

$$\therefore \sin^{-1}(\omega x)/\sqrt{2B} = \omega t + \phi_1$$

$$\text{or} \quad (\omega x)/\sqrt{2B} = \sin(\omega t + \phi_1)$$

$$x = (\sqrt{2B}/\omega)\sin(\omega t + \phi_1)$$

$$\text{put} \quad \sqrt{2B}/\omega = A_1$$

$$\text{so} \quad x = A_1 \sin(\omega t + \phi_1)$$

$$x = (-A_1)\{-\sin(\omega t + \phi_1)\}$$

$$\text{put} \quad -A_1 = A \;\;\&\;\; \phi_1 = \phi - \Pi/2$$

$$\therefore \quad x = A \sin -(\omega t + \phi - \Pi/2)$$

$$= A \sin\{\pi/2 - (\omega t + \phi)\} \qquad [\sin(\pi/2 - \theta) = \cos\theta$$

$$= A\cos(\omega t + \phi) \qquad .....(1.6)$$

which is the general solution of eq (1.3)

In eq (1.6) A is maximum amplitude

ϕ is phase constant

$(\omega t + \phi)$ is phase angle

& x is the position of the particle at time 't'

For particular case, when mass attached to a spring, having maximum amplitude, x_m, eq (1.6) modifies as:

$$x = x_m \cos(\omega t + \phi) \qquad .....(1.7)$$

differentiating w.r.t., 't', we get

$$dx/dt = v = -\omega x_m \sin(\omega t + \phi) \qquad .....(1.8)$$

differentiating once again,

$$d^2x/dt^2 = a = -\omega^2 x_m \cos(\omega t + \phi) \qquad .....(1.9)$$

Eqs (1.7), (1.8) & (1.9) are equations of **displacement, velocity** and **acceleration** of a simple harmonic oscillator.

1-3: ENERGY CONSIDERATION IN SHM

We have, kinetic energy, K

$$K = \tfrac{1}{2} m v^2$$

Putting value of 'v' from eq (1.8), we get

$$K = \tfrac{1}{2} m \{-\omega x_m \sin(\omega t + \phi)\}^2 \qquad [\omega^2 = k/m$$

$$\text{or} \quad K = \tfrac{1}{2} m \omega^2 x_m^2 \sin^2(\omega t + \phi) \qquad [\text{or } m\omega^2 = k$$

$$\text{or} \quad K = \tfrac{1}{2} k x_m^2 \sin^2(\omega t + \phi) \qquad .....(1.10)$$

Also we have, mechanical potential energy, U

$$U = \tfrac{1}{2} k x^2$$

Putting value of 'x' from eq (1.7)

$$U = \tfrac{1}{2} k \{x_m \cos(\omega t + \phi)\}^2$$

$$U = \tfrac{1}{2} k x_m^2 \cos^2(\omega t + \phi) \qquad .....(1.11)$$

From eqs (1.10) & (1.11), we have

$$K_{max} = \tfrac{1}{2} k x_m^2 \qquad .....(1.12)$$

$$U_{max} = \tfrac{1}{2} k x_m^2 \qquad .....(1.13)$$

And total mechanical energy, E

$$E = K + U = \tfrac{1}{2} k x_m^2 \{\sin^2(\omega t + \phi) + \cos^2(\omega t + \phi)\}$$

$$[\sin^2\theta + \cos^2\theta = 1$$

$$\text{or} \quad E = \tfrac{1}{2} k x_m^2 \qquad .....(1.14)$$

Eqs (1.12), (1.13) & (1.14) show that total mechanical energy is same at each point.

1-4: TIME PERIOD & FREQUENCY IN SHM

We have applied force from Newton's 2nd Law

$$F = m\,a \quad .....(1.15)$$

And restoring force from Hooke's Law

$$F = -k\,x \quad .....((1.16)$$

From eqs ((1.15) & (1.16) , we get

$$m\,a = -k\,x$$

or $$m\,d^2x/dt^2 + k\,x = 0$$

$$d^2x/dt^2 + (k/m)\,x = 0 \quad .....(1.17)$$

put $$k/m = \omega^2 \quad \text{or} \quad \omega = \sqrt{k/m} \quad .....(1.18)$$

$$d^2x/dt^2 + \omega^2 x = 0$$

whose solution we have calculated in Section 1-2, as

$$x = A\cos(\omega t + \phi) \quad .....(1.19)$$

Now from the definition of angular velocity, ω

$$\omega = \theta/t$$

for one complete cycle, time is its time period and angle is 2Π radians, so

$$\omega = 2\pi/T$$

or $$T = 2\pi/\omega \quad .....(1.20)$$

from eqs (1.18) & (1.20), we have

$$T = 2\pi/(\sqrt{k/m})$$

or $$T = 2\pi(\sqrt{m/k}) \quad .....(1.21)$$

Which is the **time period** in case of SHM.

And the **frequency** will be

$$\nu = 1/T = (1/2\pi)(\sqrt{k/m}) \quad .....(1.22)$$

We will consider **applications of SHM** in next Sections.

1-5: TORSIONAL OSCILLATOR

Definitions:

Torque (τ): A turning force or moment.

Torsion: Angular strain produced by applying torque or twisting force.

Torsional wave: A wave motion in which the vibrations in the medium are rotatory simple harmonic motions around the direction of energy transfer.

Torsion balance: A device for measuring very small forces by the torsion they cause in a wire or fiber.

Torsional oscillator: An oscillator in which the oscillations are rotatory.

Moment of Inertia: The rotational analogue of mass. The moment of inertia of an object rotating about an axis is given by ; $I = \Sigma m r^2$

Linear and their angular counter parts:

Linear quantity		Angular quantity	
$x = s$	$\rightarrow$	θ	$s = r\theta$
v	$\rightarrow$	ω	$v = r\omega$
a	$\rightarrow$	α	$a = r\alpha$
F	$\rightarrow$	τ	$\theta = \omega t$
m	$\rightarrow$	I	

Now from Hooke's Law, we have

$$F = k x$$

Its angular counter part would be,

$$\tau = -\kappa \theta \qquad \text{.....(1.23)}$$

We have from Newton's 2nd Law

$$F = m a$$

Its angular counter-part is,

$$\tau = I \alpha = I\, d^2\theta /dt^2 \qquad \text{.....(1.24)}$$

from eqs (1.23) & (1.24) , we get

$$-\kappa \theta = I\, d^2\theta /dt^2$$

or

$$d^2\theta /dt^2 = -(\kappa/I)\, \theta$$

$$d^2\theta /dt^2 + (\kappa/I)\, \theta = 0$$

following linear part from eqs (1.5) & (1.6) , we have angular counter-part of the solution of above equation as,

$$\theta = \theta_m \cos (\omega t + \phi)$$

We have time period in linear part is,

$$T = 2\pi \sqrt{(m/k)}$$

And so its angular counter-part will be

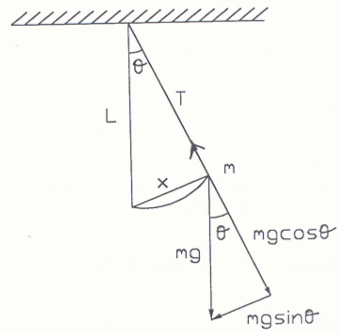
$$T = 2\pi \sqrt{(I/\kappa)} \qquad \text{.....(1.25)}$$

1-6: SIMPLE PENDULUM

Definition: *A single isolated particle suspended from a frictionless support by a light, inextensible string.*

Consider a simple pendulum swinging in a vertical plane (in 2 dimensions).

In the fig., resolving force mg (weight of the particle) into two components. one along its radius and the other along the tangent of the circular path.



Radial force is providing centripetal force necessary to hold the particle, and tangential force is responsible for its motion along circular path.

Radial force: $F = mg\cos\theta$

& tangential force: $F = mg\sin\theta$

and restoring force responsible for its backward motion,

$$F = -mg\sin\theta$$
$$F = -mg\,\theta$$
$$F = -mg\,(x/L)$$
$$F = -(mg/L)\,x \qquad .....(1.26)$$

Taylor's series expansion for,
$\sin\theta = \theta - \theta^3/3! + \theta^5/5! - ..$
for small θ, $\sin\theta = \theta$
from the fig.
$\sin\theta = \theta = x/L$

In eq (1.26) m, g & L are constant for a particular case, therefore

$$F \propto -x$$

or $$F \propto -(\text{displacement})$$

Which is the characteristic of Simple Harmonic Motion.
Therefore, **simple pendulum executes SHM**.

To calculate time period of simple pendulum,
Comparing eqs (1.16) & (1.26), we have

$$k = mg/L \qquad .....(1.27)$$

from eqs (1.21) & (1.27), we get

$$T = 2\pi\sqrt{(m\,L/mg)}$$
$$T = 2\pi\sqrt{(L/g)} \qquad .....(1.28)$$

Which is the **time period** of simple pendulum.
We see that

$$T \propto \sqrt{L}$$
$$\propto \sqrt{1/g}$$

and does not depend upon mass, m of the particle.

1-7: PHYSICAL PENDULUM

Definition: *Any rigid body mounted so that it can swing in a vertical plane about some axis passing through it.*

Inertia: 1) It is a property by virtue of which it is necessary to exert a force on a body at rest if it is to be set into motion.
2) The resistance of matter to any acceleration of its state of rest or motion.
3) An inherent property of matter implied by Newton's first law of motion;

Inertial mass: The mass of an object as measured by the property of inertia; Mathematically, it is equal to , $m = F/a$, for constant force F.

Gravitational mass: The mass of the body as measured by the force of attraction between masses, the value being given by Newton's law of gravitation.

$$F = G(m_1 m_2 / r^2)$$

Relativistic mass: The mass of an object as measured by an observer at rest in a frame of reference in which the object is moving with velocity v; It is given by

$$m = m_o(\sqrt{(1 - v^2/c^2)})$$

We have

$$F = m g$$

$$\& \quad \vec{\tau} = \vec{F} \times \vec{r} = F d \sin\theta \qquad \left\{ \begin{array}{l} \text{for small } \theta \\ \sin\theta = \theta \end{array} \right.$$

$$\text{or} \quad \tau = -M g d \sin\theta$$

$$\tau = -M g d \theta \qquad \text{.....(1.29)}$$

We have from Hooke's Law,

$$F = -k x$$

Its rotational counter-part (see Sect 1-5), will be

$$\tau = -\kappa\theta \qquad \text{.....(1.30)}$$

Comparing eqs (1.29) & (1.30), we get

$$\kappa = M g d \qquad \text{.....(1.31)}$$

Taking

$$T = 2\pi\sqrt{(m/k)} \qquad [\omega = \theta / t \text{ or } t = \theta / \omega = 2\pi / \sqrt{m/k}$$

In rotational terminology, we have

$$T = 2\pi\sqrt{(I/\kappa)} \qquad \text{.....(1.32)}$$

From eqs (1.31) & (1.32) we have

$$T = 2\pi\sqrt{(I/Mgd)} \qquad \text{.....(1.33)}$$

$$\text{or} \quad I = T^2 Mgd / 4\pi^2 \qquad \text{.....(1.34)}$$

The physical pendulum includes the simple pendulum as a special case. Locating the pivot far from the object, using a weightless chord of length L, we have then

From eq (1.33),

$$T = 2\pi\sqrt{(ML^2/MgL} \qquad \left\{ \begin{array}{l} I = m r^2 \\ I = M L^2 \ \& \ d = L \end{array} \right.$$

$$\text{or} \quad T = 2\pi\sqrt{(L/g)} \qquad \text{.....(1.35)}$$

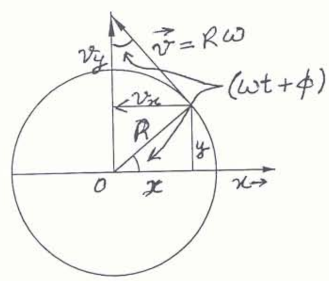
From eqs (1.33) & (1.35) we have

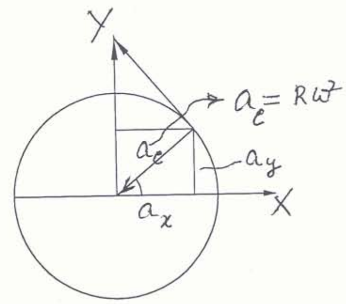
$$L/g = I/Mgd$$

$$\text{or} \quad L = I/Md \qquad \text{.....(1.36)}$$

This point whose distance from the pivot is 'L' given by above equation, is called the center of oscillation of the physical pendulum.

1-8: SHM & CIRCULAR MOTION





From the fig. we have

$$x = R\cos(\omega t + \phi) \qquad .....(1.37)$$

$$y = R\sin(\omega t + \phi) \qquad .....(1.38)$$

$$y = R\cos\{\pi/2 - (\omega t + \phi)$$

$$y = R\cos -(\omega t + \phi - \Pi/2)$$

$$y = R\cos(\omega t + \phi - \Pi/2) \qquad .....(1.39)$$

$$\left[\begin{array}{l} \cos(\pi/2 - \theta = \sin\theta \\ \cos(-\theta) = \cos\theta \end{array} \right.$$

From eqs (1.37) & (1.38) we have

$$x^2 + y^2 = R^2\{\cos^2(\omega t + \phi) + \sin^2(\omega t + \phi)\}$$

$$\text{or } x^2 + y^2 = R^2 \qquad .....(1.40)$$

Eq (1.40) is equation of a circle, as the particle is executing circular motion. Also eqs (!.37) & (1.39) show that circular motion is combination of two SHM at right angles.

From the fig. we have $\qquad [\, v = r\,\omega$

$$v_x = -R\,\omega\sin(\omega t + \phi) \qquad .....(1.41)$$

$$[\, a_c = v^2/r = r^2\omega^2/r = r\,\omega^2$$

$$\& \qquad a_x = -\omega^2 R(\cos\omega t + \phi) \qquad .....(1.42)$$

from eqs (1.38) , (1.41) & (1.42) we see that x , v & a are identical in SHM and in the projection of circular motion.

Also we have from the fig. $\qquad$ [Compare eqs. (1.7), (1.8) & (1.9)

$$v_y = -R\,\omega\cos(\omega t + \phi) \qquad .....(1.43)$$

$$a_y = -\omega^2 R\sin(\omega t + \phi) \qquad ....(1.44)$$

from eqs (1.41) & (1.43) we have

$$v_x^2 + v_y^2 = (\omega R)^2 \qquad .....(1.45)$$

from eqs (1.42) & (1.44) we have

$$a_x^2 + a_y^2 = (\omega^2 R)^2 \qquad .....(1.46)$$

The results of above equations show that SHM can be described as the projection of uniform circular motion along a diameter of the circle.

1-9: COMBINATIONS OF HARMONIC MOTIONS

Consider two SHMs having different amplitudes and phase constants.

$$x = x_m \cos(\omega t + \phi_x) \quad .....(1.47)$$

$$y = y_m \cos(\omega t + \phi_y) \quad .....(1.48)$$

If phase constants are same, i.e., $\phi_x = \phi_y$

$$x = x_m \cos(\omega t + \phi_x) \quad .....(1.49)$$

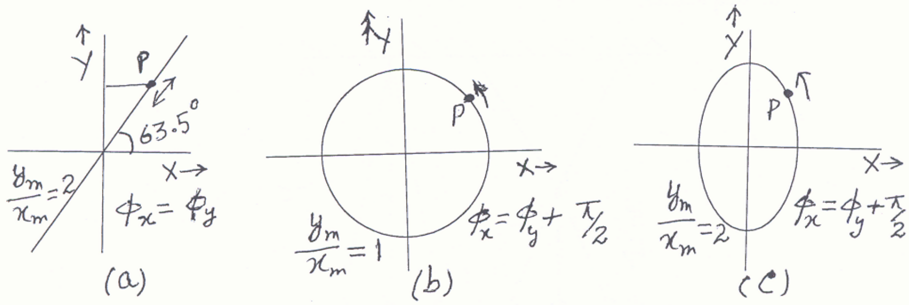
$$y = y_m \cos(\omega t + \phi_x) \quad .....(1.50)$$

dividing eq (1.50) with (1.49)

$$y/x = y_m/x_m$$

$$\text{or} \quad y = (y_m/x_m)\, x \quad .....(1.51)$$

Eq (1.51) is equation of straight line having slope (y_m/x_m) [see Appendix-C]
If $(y_m/x_m) = 2$ or $y_m = 2\,x_m$, then we get the following line:



If phase constants are different but amplitudes are same,
i.e. $(y_m/x_m) = 1$ or $y_m = x_m$ & $\phi_x = \phi_y + \pi/2$
then from eqs (1.47) & (1.48) we have

$$x = x_m \cos(\omega t + \phi_y + \pi/2) \qquad [\cos(\theta + \pi/2) = -\sin\theta$$

$$\text{or} \quad x = x_m \{-\sin(\omega t + \phi_y)\} \quad .....(1.52)$$

$$\& \quad y = x_m \cos(\omega t + \phi_y) \quad .....(1.53)$$

squaring (1.52) & (1.53) then adding, we get

$$x^2 + y^2 = x_m^2 \quad .....(1.54)$$

which is equation of a circle. [see Appendix-C]

Now if phase constants & amplitudes (both) are different
i.e.

$$y_m = 2\,x_m \quad \& \quad \phi_x = \phi_y + \pi/2$$

then from eqs (1.47) & (1.48) we have

$$x = x_m \cos(\omega t + \phi_y + \pi/2)$$

$$\text{or} \quad x = x_m \{-\sin(\omega t + \phi_y)\}$$

$$\& \quad y = 2x_m \cos(\omega t + \phi_y)$$

squaring will give

$$x^2 = x_m^2 \sin^2(\omega t + \phi_y) \quad .....(1.55)$$

$$y^2 = 4x_m^2 \cos^2(\omega t + \phi_y) \quad .....(1.56)$$

from eqs (1.55) & (1.56) we get

$$x^2/x_m^2 + y^2/4x_m^2 = 1 \quad .....(1.57)$$

which is equation of an ellipse [see Appendix-C]

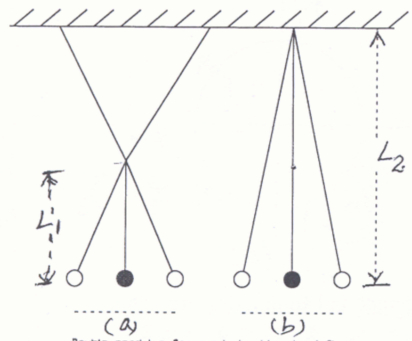
1-10: LISSAJOUS' PATTERNS

Definition: *The curves made by the paths traced out by a particle which is oscillating simultaneously in two mutually perpendicular directions.*

In these patterns, the amplitude, frequency and phase constant can be different.

Example:

1. In the following figure a pendulum bob is supported by three cords. When it vibrate in X-direction (a), its length is L_1. And when it vibrates in Y-direction (b), the length of the pendulum is L_2. If displaced in both x- and y-directions, the bob executes vibrations of both frequencies.



Double pendulum for producing Lissajous' figures

2. In Cathode-ray oscilloscope (CRO), if sinusoidal alternating voltages are applied simultaneously to the horizontal and vertical deflecting plates, the spot on the screen produced by the impact of a rapidly moving stream of electrons will also move in a Lissajous figure.

Qualitative Analysis:

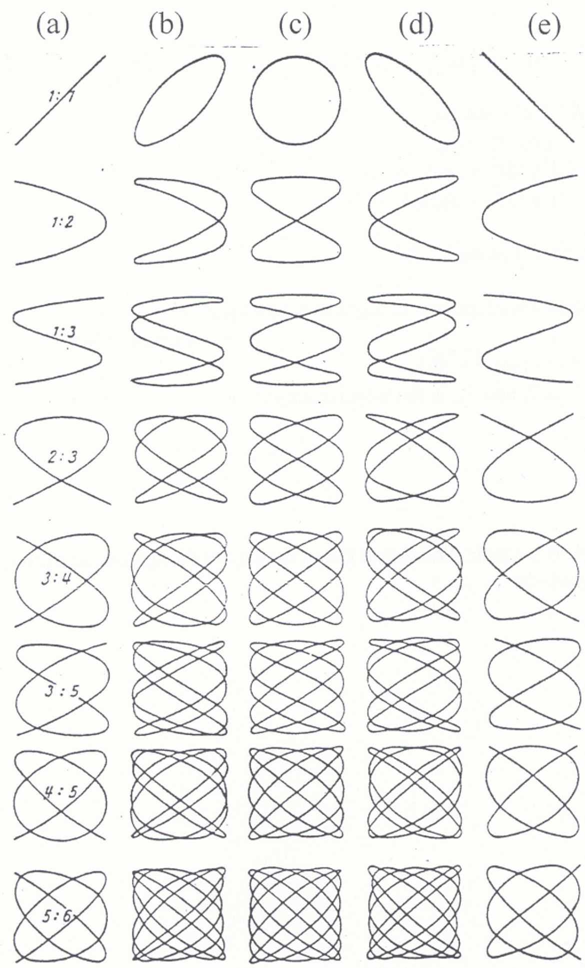
The equations for x- and y-coordinates for the oscillating system are

$$x = R_x \cos(\omega_x t + \phi_1) \qquad \text{.....(1.58)}$$

$$y = R_y \sin(\omega_y t + \phi_2) \qquad \text{.....(1.59)}$$

where R_x & R_y are respective amplitudes, ω_x & ω_y are corresponding angular frequencies, and ϕ_1 & ϕ_2 are phase constants.

Some Lissajous figures are shown in the fig. for various frequency ratios and initial phase differences ($\phi_2 - \phi_1$). The amplitudes R_x and R_y are equal in each case. If the frequencies are reducible to common measure, the particle retraces a closed path over and over. If they are not, the path does not close on itself, and the pattern may be extremely complicated. If the frequencies are very nearly equal, the path changes slowly from a straight line at 45°, as in fig.(a), to an ellipse as in (b), then to a circle as in (c), then to an ellipse as in (d), with its major axis at right angles to that in (b), then to a straight line as in (e), and so on.



1-11: ANGULAR HARMONIC MOTION

Mathematical analog of linear harmonic motion is angular harmonic motion.
Let a body be pivoted about a fixed axis and acted on by a restoring torque 'M' proportional to the angular displacement 'ϕ' from some reference position, then

$$M = -\kappa\phi \qquad .....(1.60)$$

where κ is called torque constant

[linear analog: $F = -kx$]

Also we have

$$M = I\alpha \qquad .....(1.61)$$

[$\tau = I\alpha$

From eqs. (1.60) & (1.61), we get

$$I\alpha = -\kappa\phi$$

$$I\, d^2\phi/dt^2 + \kappa\phi = 0$$

or $$d^2\phi/dt^2 + (\kappa/I)\phi = 0 \qquad .....(1.62)$$

eq. (1.62) has same form as eq. (1.3).

By analogy, the equation of motion in angular terminology is

[$x = x_m \sin(\omega t + \phi)$

$$\phi = \phi_m \sin(\omega t + \theta_o) \qquad .....(1.63)$$

where $\omega = \sqrt{\kappa/I}$, and ϕ_m is the angular amplitude.

Example:

The balance wheel of a watch. The watch keeps time even though the amplitude decreases as the main spring unwinds.

1-12: DAMPED HARMONIC MOTION

We have

restoring force : $F = -k x$ .....(1.64)

Assume that the object of mass 'm' experiences a damping force that increases linearly with velocity 'v'

$$F = -b v \qquad .....(1.65)$$

[negative sign shows that damping force is opposite to the direction of velocity]

from Newton's 2nd Law

$$\Sigma F = m a \qquad .....(1.66)$$

from eqs. (1.64) , (1.65) & (1.66), we get

$$-k x - b v = m a$$

$$\text{or} \quad -k x - b\, dx/dt = m\, d^2x/dt^2$$

$$\text{or} \quad m\, d^2x/dt^2 + b\, dx/dt + kx = 0$$

$$\text{or} \quad d^2x/dt^2 + (b/m)(dx/dt) + (k/m)x = 0$$

$$\text{or} \quad (D^2 + b/m\, D + k/m)x = 0 \qquad .....(1.67)$$

$$[\, d/dt = D, \quad d^2/dt^2 = D^2 \,]$$

from eq. (1.67), the auxiliary equation is

$$D^2 + (b/m) D + k/m = 0$$

Its solution is

$$D = \frac{-b/m \pm \sqrt{(b^2/m^2) - 4(1)(k/m)}}{2(1)} \qquad \left[x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \right.$$

$$D = \frac{-b/m \pm \sqrt{-4(k/m - (b/2m)^2}}{2}$$

$$\text{put} \quad \sqrt{k/m - (b/2m)^2} = \omega'$$

$$\text{so} \quad D = \frac{-b/2m \pm i\, 2\omega'}{2}$$

$$\text{or} \quad D = -b/2m \pm i\,\omega'$$

[As the solution of $D = a \pm i b$ is $x = e^{at}\{A \cos bt + B \sin bt\}$]

$$\therefore \quad x = e^{-bt/2m}\{A \cos \omega' t + B \sin \omega' t\}$$

put $A = x_m \cos\phi$ & $B = x_m \sin\phi$

then

$$x = e^{-bt/2m}\{x_m \cos\phi \cos\omega' t + x_m \sin\phi \sin\omega' t\}$$

$$x = x_m e^{-bt/2m}\{\cos\omega' t \cos\phi + \sin\omega' t \sin\phi\}$$

[since $\cos(\alpha + \beta) = \cos\alpha \cos\beta - \sin\alpha \sin\beta$, therefore,]

$$x = x_m e^{-bt/2m} \cos(\omega' t + \phi), \text{ where } \omega' = \sqrt{k/m - (b/2m)^2} \qquad .....(1.68)$$

and $\quad \text{amplitude} = x_m e^{-bt/2m}$ .....(1.69)

Two features to note from eqs. (1.68) & (1.69),

i) with friction 'b' , frequency 'f' is less (or time period 'T' is longer) , so friction slows down the motion

ii) curve between displacement 'x' and 't' decreases exponentially to zero.

The equations of damped harmonic motion:

1) Under damped ($b < 2\sqrt{km}$)

$$x = A e^{-\alpha t} \cos(\omega' t + \phi) \quad \text{....(1.70)}$$

where $\alpha = b/2m$ & $A = x_m$

{ for un-damped
$x = x_m \cos(\omega t + \phi)$,
$\omega = \sqrt{k/m}$

2) Critically damped ($b = 2\sqrt{km}$)

If $b = 0$, then from eq. (1.68) , $\omega' = 0$

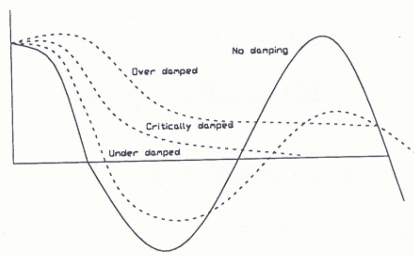
So $x = (B_1 + B_2 t) e^{-\beta t}$, $\beta = \sqrt{k/m}$ .....(1.71)

And displacement approaches to zero exponentially with no oscillations. And mean life time 'τ' has smallest value.

3) Over damped ($b > 2\sqrt{km}$)

$$x = C_1 e^{-\gamma_1 t} + C_2 e^{-\gamma_2 t} \quad \text{.....(1.72)}$$

Graph of simple and damped harmonic motion



In the fig. the full curve shows the motion when the damping factor is zero. For over damping, the period is greater than in the absence of damping and the amplitude of successive oscillations becomes smaller and smaller. For over damping, the body returns even more slowly to its equilibrium position. For critical damping, the motion ceases to be oscillatory, and the body returns to its equilibrium position without over-shooting, which is the goal of mechanical engineers in designing a system in which the oscillations disappear in the shortest possible time.

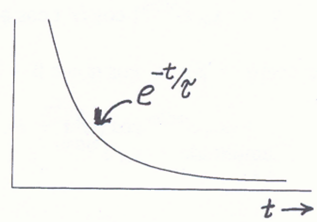
Mean life time (τ) :

The time interval during which the amplitude drops to 1/e of its initial value.

$e^{-bt/2m} = e^{-1}$; when $t = 2m/b$

for $b = 0$,

x_m will be constant and life time would be infinite.



1-13: FORCED OSCILLATIONS & RESONANCE

Assume that a system is acted on by a driving force which varies sinusoidally with time according to the relation

$$F = F_m \cos \omega'' t \qquad .....(1.73)$$

Also we have the restoring force

$$F = -k x \qquad .....(1.74)$$

& damping force

$$F = -b v \qquad .....(1.75)$$

From Newton's 2nd Law of motion

$$\Sigma F = m a$$

$$\text{or} \quad F_m \cos\omega'' t - k x - b v = m a \qquad .....(1.76)$$

$$\text{or} \quad m\, d^2x/dt^2 + b\, dx/dt + k x = F_m \cos \omega'' \qquad .....(1.77)$$

the solution of eq. (1.77) is

$$x = (F_m /G) \sin (\omega'' t - \phi) \qquad .....(1.78)$$

$$\text{where} \quad G = \sqrt{m^2(\omega''^2 - \omega^2)^2 - b^2\omega''^2} \qquad .....(1.79)$$

$$\& \quad \phi = \cos^{-1} b\, \omega'' /G \qquad .....(1.80)$$

by calculating dx/dt & d^2x/dt^2 in eq. (1.78), it can be verified that this equation is a solution of the differential eq. (1.77).

from eq. (1.79), if $b = 0$

$$G = m(\omega''^2 - \omega^2)$$

This means that 'G' is large for ω'' different from ω.

And from eq. (1.78),

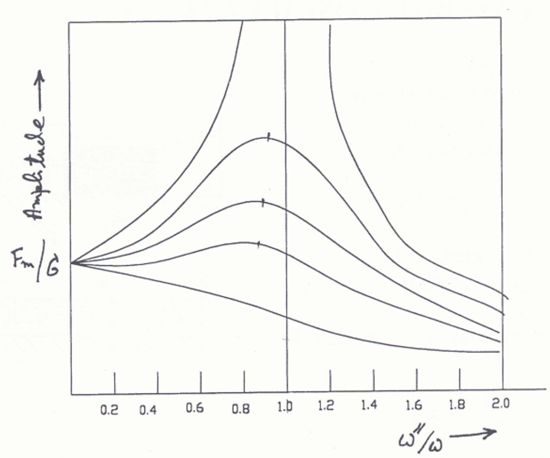
$\Rightarrow$ amplitude F_m /G is small

As driving frequency $\omega'' \rightarrow$ natural frequency ω

$$G \rightarrow 0$$

$$\& \; F_m /G \rightarrow \infty$$

In actual system, some damping is always present and amplitude F_m /G does not go to infinity.



The graph on the last page in the fig. is for the idealized case of a frictionless spring-mass system with an infinite proportional limit of elasticity. The amplitude equals
F_m/k when $\omega'' = \omega$

$$\left\{ \begin{array}{l} \omega = \sqrt{k/m} \\ k = m\,\omega^2 \end{array} \right.$$

When damping is present, provided it is less than critical, the amplitude passes through a maximum as frequency is varied. This phenomenon is known as **resonance.** It is shown by next three curves in fig. The maximum amplitude does not occur at frequency
$\omega'' = \omega$ [see eq. (1.79)]

The value of ω'' at which resonance occurs is called **resonant angular frequency.**
If a system is over-damped, the graph of amplitude verses ω'/ω does not pass through a maximum and the amplitude decreases steadily with increasing frequency. This is shown by the lowest curve in fig.

Illustrations:
The forced oscillations have the frequency of the external force and not the natural frequency of the body. The response of the body depends on the relation between the forcing and the natural frequencies. A succession of small impulses applied at the proper frequency can produce an oscillation of large amplitude.
All mechanical structures—such as buildings, bridges and air planes—have one or more natural frequencies. It can be disastrous to subject to the structure to an external driving force at one of those frequencies.

1. A child using a swing learns to pump at proper time intervals to make the swing move with a large amplitude.

2. The image of a soprano shattering a cup.

3. The wind blowing through Tacoma Narrows (Washington, U.S.A.) shook the bridge at a frequency that matched one of its natural frequencies.

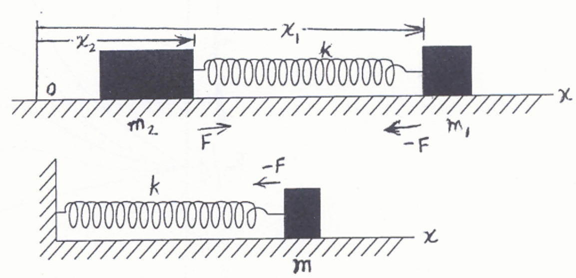
1-14 TWO BODY OSCILLATIONS

Consider separate motions of the two particles, shown in the fig. Their relative distances from origin O are x_1 and x_2
The relative separation is
$(x_1 - x_2)$
unstretched length is L.
so, change in length is
$(x_1 - x_2) - L$
Force exerted on each particle is F:

$$F = k\,x$$

or $m\,a = k\,x$

or $m\,d^2x/dt^2 = k\,x$



for particle having mass m_1, we have

$$m_1 \, d^2x_1 /dt^2 = -k\,x \quad .....(1.81)$$

for particle having mass m_2, we have

$$m_2 \, d^2 x_2 / dt^2 = +k\,x \quad .....(1.82)$$

multiplying eq. (1.81) by m_2 & eq. (1.82) by m_1 , we get

$$m_1 m_2 \, d^2 x_1 /dt^2 = -m_2 \, k\,x \quad .....(1.83)$$

$$m_1 m_2 \, d^2 x_2 / dt^2 = +m_1 \, k\,x \quad .....(1.84)$$

subtracting eq. (1.83) from eq. (1.84), we get

$$m_1 m_2 \{d^2 /dt^2 \, (x_1 - x_2)\} = -kx \, (m_1 + m_2)$$

$$m_1 m_2 /(m_1 + m_2) \, \{d^2 x /dt^2 \, (x_1 - x_2)\} = -k\,x \quad .....(1.85)$$

We define, reduced mass, m, as

$$m = m_1 m_2 /(m_1 + m_2) \quad .....(1.86)$$

& relative displacement, x, as

$$x = x_1 + x_2 \quad .....(1.87)$$

from eqs. (1.85), (1.86) & (1.87), we get

$$m \, d^2 x /dt^2 = -k\,x$$

or

$$d^2 x /dt^2 + (k/m)\, x = 0 \quad .....(1.88)$$

which is same as eq. (1.3).

therefore system of fig. (a) can be replaced by a single particle as shown in fig. (b)

Characteristics

1. The reduced mass , m, is always smaller than either mass.
   e. g. $m = (2 \times 2) /(2 + 2) = 4/4 = 1$
2. If one of the masses is very much smaller than the other, the m is roughly equal to smaller mass.
   e. g. $m = (1 \times 100) /(1 + 100) \approx 1$
3. If the masses are equal, then m is half as large as either mass.
   e. g. $m = (10 \times 10) /(10 + 10) = 100/20 = 5$

****

WAVE MOTION

2-1: DEFINITIONS

Wave motion: The mechanism by which energy is transferred form one particle to another particle.

Wave: A disturbance in the medium.

Medium: Any material---solid, liquid or gas in which waves travel.

Wave pulse: Single unrepeated disturbance.

Mechanical waves: The wave which require a medium for their movement.

Electromagnetic waves: Transverse waves in space having an electric component and a magnetic component.

Matter waves: These waves carry energy and pilot the particle and move along with it.

Travelling waves: Waves produced by a driving force, and they travel away from the source which produces them.

Stationary waves (or Standing waves): 1) The resultant of two wave trains of the same wavelength, frequency and amplitude travelling in opposite directions through the same medium.
2) Waves apparently standing still resulting from two similar wave trains travelling in opposite directions.

Transverse wave: A wave in which the particle of the medium vibrate at right angles to the direction of travel of the wave.

Longitudinal wave (or Compressional wave): A wave in which the particles of the medium vibrate to and fro parallel to the direction of travel of the wave.

Light: The aspect of radiant energy of which an observer is visually aware.

Ray: A single line of light coming from a luminous point.

Luminous: Objects that give off light of their own.

Beam: Several parallel rays of light considered collectively.

Wave front: Locus of all points having the same phase of vibration.

Spherical (circular) wave front : In homogeneous medium, from a point source , concentric spheres (circles) of wave fronts with center at the source.

Plane wave front: A small portion of a spherical wave front which very nearly plane.

2-2 : TRAVELLING WAVES

Travelling waves produced by a driving force, and they travel away from the source. They have the property of transporting energy and momentum from one point to another. If a generator oscillates with simple harmonic motion and is connected to a string will produce travelling transverse waves.
Consider a single pulse of harmonic waves travelling along the string.

Fig 2-1:

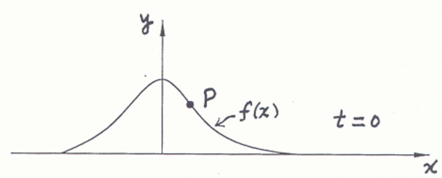
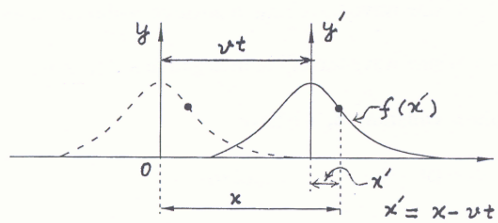


Fig 2-2:



The waveform f(x) in fig 2-1 depends on x and t , i.e.,

$y(x,0) = f(x)$ ......(2.1) [$y = f(x)$

$y(x,t) = f(x') = f(x - vt)$ .....(2.2)

To keep same shape of wave form, for the motion of any particular phase of the wave,

$(x - vt) = \text{constant}$

Differentiating w.r.t. t, we have

$dx/dt - v = 0$

or $dx/dt = v$

where v is called **phase velocity**

For Sinusoidal waves, take a transverse wave along a string at t = 0 , we have

$y(x,0) = y_m \sin k x$ ....(2.3)

$y = \sin\theta$
$y(t) = A \sin\theta$
$y(t) = A \sin \omega t$
$y(x) = A \sin \omega x$
$y(x) = y \sin \omega x$
$y(x,0) = y \sin k x$

<u>Definition</u>: **Wave Number** or **Circular wave number , k**

The reciprocal of the wavelength of a wave times 2π ; mathematically

$$k = 2\pi/\lambda = 2\pi\nu$$

from eq. (2.3) , we have

$$y(x,0) = y_m \sin 2\pi/\lambda(x)$$

from eqs. (2.1) and (2.2) , we have

$$y(x,t) = y_m \sin 2\pi/\lambda(x - vt)$$
$$= y_m \sin(2\pi/\lambda(x) - 2\pi vt/\lambda)$$
$$= y_m \sin(kx - 2\pi t/T)$$

$$\left\{ \begin{array}{l} k = 2\pi/\lambda \\ S = vt \\ \lambda = vT \text{ or } 1/T = v/\lambda \\ \omega = 2\pi\nu = 2\pi/T \end{array} \right.$$

or $y(x,t) = y_m \sin(kx - \omega t)$ .....(2.4)

Also $y(x,t) = y_m \sin(kx + \omega t)$ .....(2.5)

Eq. (2.4) is the equation of sine wave travelling in positive x-direction, and

Eq. (2.5) is the equation of sine wave travelling in negative x-direction.

In eq (2.4) adding a **phase constant** ϕ, we have

$y(x,t) = y_m \sin(kx - \omega t - \phi)$ .....(2.6)

where ϕ is called **phase constant** and

$(kx - \omega t - \phi)$ is called **phase angle** or **phase of the wave**

re-arranging eq (2.6), we get

$y(x,t) = y_m \sin\{(kx - k\phi/k) - \omega t\}$

or $y(x,t) = y_m \sin\{k(x - \phi/k) - \omega t\}$ .....(2.7)

&

$y(x,t) = y_m \sin\{kx - (\omega t - \omega\phi/\omega)\}$

or $y(x,t) = y_m \sin\{kx - \omega(t + \phi/\omega)\}$ .....(2.8)

eqs (2.7) & (2.8) show that the phase constant does not affect the shape of the wave, it only <u>moves</u> the wave forward or backward in <u>space</u> or <u>time</u>

From eq (2.6) , we have

$y(x,t) = -y_m \sin(\omega t + \phi - kx)$ .....(2.9)

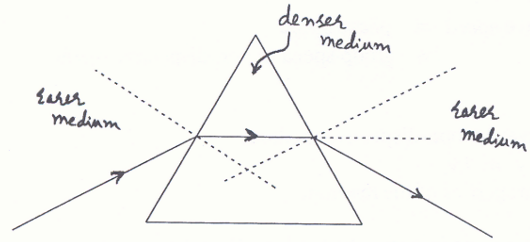
put $\phi - kx = \phi'$ in the above eq

$y(x,t) = -y_m \sin(\omega t + \phi')$ .....(2.10)

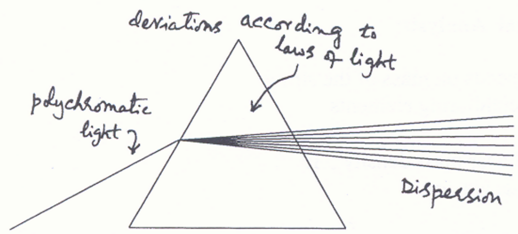
eq (2.10) shows that any particular element of the string goes <u>Simple Harmonic Motion</u> for travelling waves.

2-3: DEFINITIONS

Principle of refraction: When a ray of light enters from rarer to denser medium it bends towards the normal and if the ray of light enters from denser to rarer medium, it bends away from the normal.



fig=2-3



fig= 2-4

Dispersion: 1) An effect in which radiation's having different wavelengths travel a different speeds in the medium.
2) The separation of polychromatic light into its component wavelengths.

Dissipation: The removal of energy from a system to overcome some mechanical or electrical force.

Group Speed: It is the velocity with which the group of waves travel. If a wave motion has a phase velocity that depends on wavelength, the disturbance of a progressive wave travels with a different velocity. This is called group velocity.

Dispersive medium: A medium in which the phenomenon of dispersion occurs.

Phase speed: The speed with which the phase in a travelling wave is propagated. It is equal to λ/T, where T is the period.

$$\begin{cases} v = \lambda\nu, \quad \nu = 1/T \\ v = \lambda/T \end{cases}$$

In non-dispersive medium group speed is equal to phase speed.

2-4: WAVE SPEED

Considering sinusoidal waves , we know from our previous knowledge

Wave speed ≡ phase speed
≡ group speed in non-dispersive medium

for mechanical waves in non-dispersive medium

$$v \neq \lambda\nu$$

but v depends on properties of the medium.

Consider transverse waves in a stretched string. Calculating speed of waves from dimensional analysis and mechanical analysis.

i) **Dimensional Analysis:**

Speed of waves depends on mass of the string
& force between neighboring elements

mass corresponds linear mass density , μ
& force corresponds tension, F

$$\therefore \quad v \propto F^a$$

$$\propto \mu^b$$

or $$v \propto F^a \mu^b \qquad .....(2.11)$$

where a & b are to be determined.
Putting relation (2.11) in terms of dimensions

$$[v] = [F^a][\mu^b]$$
$$LT^{-1} = (MLT^{-2})^a (ML^{-1})^b$$
$$LT^{-1} = M^{a+b} L^{a-b} T^{-2a}$$

v = cm/sec = L/T = LT^{-1}
& F = ma = kg-m/sec^2 = MLT^{-2}

μ = mass/vol = mass/linear length = ML^{-1}

Now exponent of M : a + b = 0
exponent of L : a - b = 1
exponent of T : -2a = -1

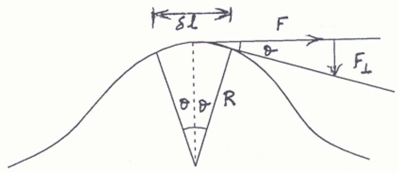
$$\Rightarrow a = ½$$
$$\Rightarrow b = ½ - 1 = -½$$
$$\therefore \quad v \propto F^{1/2}\mu^{-1/2}$$
$$v \propto \sqrt{F/\mu}$$
$$v = C\sqrt{F/\mu} \qquad .....(2.12)$$

ii) **Mechanical Analysis:**

To derive an expression for the speed of a pulse in a stretched string

Pre-requisite: Relativistic frame of reference

fig: 2-5



Horizontal component of force on the string

$F = +F\cos\theta = -F\cos\theta$

& vertical component

$F_\perp = 2F\sin\theta$

$= 2F\,\theta$

$F_\perp = F\,2\theta$

$F_\perp = F\,\delta l/R$

for small θ

$\sin\theta = \theta$

$\sin\theta = \dfrac{\delta l/2}{R}$

or $\theta = \delta l/2R$

$2\theta = \delta l/R$

density = mass/volume

$D = m/V$

linear mass density

$\mu = \delta m/\delta l$

$\delta m = \mu\,\delta l$

Equating the net vertical force
To the needed centripetal force

$F\,\delta l/R = \mu\,\delta l\,v^2/R$

$F = \mu\,v^2$

$v = \sqrt{F/\mu}$ ….. (2.13)

but from eq. (2.12)

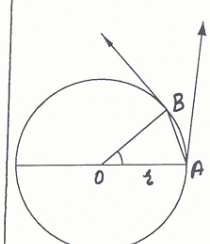
$v = C\sqrt{F/\mu}$

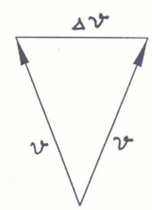
$\Rightarrow C = 1$

$F_c = m\,v^2/r = \delta m\,v^2/R$

but $\delta m = \mu\,\delta l$

$\therefore F_c = \mu\,\delta l\,v^2/R$





For small θ

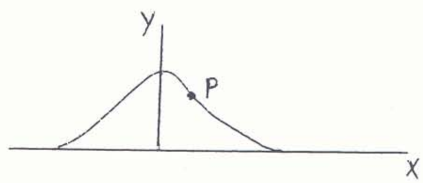
$\dfrac{AB}{r} = \dfrac{\Delta v}{v}$

$\dfrac{v\Delta t}{r} = \dfrac{\Delta v}{v}$

$\dfrac{\Delta v}{\Delta t} = a_c = \dfrac{v^2}{r}$

2-5 : PARTICLE SPEED

Calculating transverse velocity of a particle of the medium, in which tranverse sinusoidal travelling wave is travelling. Notice that wave speed or group speed is speed of waves not of the medium.



Consider single particle of the string, which is on a certain position on X-axis, so we need partial derivative of y w.r.t. t at constant x (see Appendix-D), we have

$$y(x,t) = y_m \sin(kx - \omega t - \phi)$$

or $\partial y/\partial t = -y_m \omega \cos(kx - \omega t - \phi)$

& $\partial^2 y/\partial t^2 = -y_m \omega^2 \sin(kx - \omega t - \phi)$

or $\partial^2 y/\partial t^2 = a = -\omega^2 y$

$\Rightarrow a \propto -y$

$x = x_m \cos(\omega t + \phi)$
$y = y_m \sin(kx - \omega t - \phi)$

$d/dx \sin x = \cos x$
$d/dx \cos x = -\sin x$

which is the characteristic of <u>Simple Harmonic Motion</u>, therefore, each particle of the string undergoes tranverse simple harmonic motion as the sinusoidal wave passes.

2-6 : WAVE EQUATION

We have

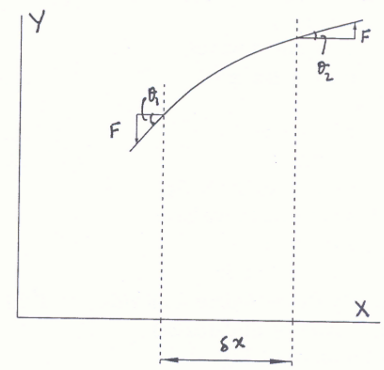
$F = -kx \quad \& \; F = ma$

or $\quad ma = -kx$

or $\quad d^2x/dt^2 = -(k/m)\,x$ ......(2.14)

whose solution is eq. (1.7).

Now finding solution for eq. (2.2)



From the figure we have

Net force in the Y-direction

$F_y = F\sin\theta_2 - F\sin\theta_1$
$\quad = F\tan\theta_2 - F\tan\theta_1$
$\quad = F(\tan\theta_2 - \tan\theta_1)$
$F_y = F\,\delta\tan\theta$ ..... (2.15)

$\sin\theta = \theta - \theta^3/3! + \theta^5/5! - \ldots\ldots$
$\tan\theta = \theta + \theta^3/3 + 2\theta^5/15 + \ldots\ldots$
for small θ
$\sin\theta = \tan\theta$

From Newton's 2nd Law
$F = ma$
In this particular case
$F_y = \delta m\, a_y$
Or $F_y = \mu\,\delta x\, a_y$ .....(2.16)
From eqs (2.15) & (2.16), we get
$F\,\delta\tan\theta = \mu\,\delta x\, a_y$

density = mass/Volume
$D = m/V$
linear mass density
μ = mass/unit length
$\mu = \delta m/\delta x$
$\delta m = \mu\,\delta x$

$\delta(\tan\theta)/\delta x = \mu/F\,(a_y)$

$$\frac{\delta(\partial y/\partial x)}{\delta x} = \frac{\mu}{F}\,a_y$$

$\delta/\delta x(\partial y/\partial x) = \mu/F\,(a_y)$ .....(2.17)
Now
in the limiting case
$\mathrm{Lim}_{\delta x\to 0}\,\delta/\delta x(\partial y/\partial x) = \partial/\partial x\,(\partial y/\partial x) = \partial^2 y/\partial x^2$ .....(2.18)
From eq (2.17) & (2.18)
$\partial^2 y/\partial x^2 = \mu/F(a_y)$
$\partial^2 y/\partial x^2 = \mu/F(\partial^2 y/\partial t^2)$ .....(2.19)

from eq (2.13)
$v = \sqrt{F/\mu}$
$v^2 = F/\mu$ or $\mu/F = 1/v^2$

$a = d^2x/dt^2 = \partial^2 y/\partial t^2$

from eqs (2.13) & (2.19), we get

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} \qquad .....(2.20)$$

Eq. (2.20) is called **Wave Equation.**

2-7: DERIVATION OF WAVE SPEED (Alternate method)

We have
for travelling waves
$y(x,t) = f(x \pm vt)$ .....(2.21)
Let
$z = x \pm vt$ .....(2.22)
$y = f(z)$ .....(2.23)
so $\partial y/\partial x = df/dz(\partial z/\partial x)$
from eq (2.22), we have
$\partial z/\partial x = 1$
$\therefore \partial y/\partial x = df/dz$
& $\partial^2 y/\partial x^2 = d/dz(df/dz)\partial z/\partial x = d^2f/dz^2$ .....(2.24)
also $\partial y/\partial t = df/dz(\partial z/\partial t)$
but from eq (2.22)
$\partial z/\partial t = \pm v$
$\therefore \partial y/\partial t = \pm v\, df/dz$
& $\partial^2 y/\partial t^2 = d/dz(\pm v\, df/dz)\partial z/\partial t = v^2\, d^2f/dz^2$ .....(2.25)
from eqs (2.24) & (2.25), we have
$\partial^2 y/\partial x^2 = 1/v^2(\partial^2 y/\partial t^2)$ .....(2.26)
from eqs (2.19) & (2.26), we get
$\mu/F = 1/v^2$
Hence $\mathbf{v = \sqrt{F/\mu}}$ .....(2.27)

<u>Chain Rule:</u>
We have
$y = f(u)$ & $u = \psi(x)$
if
dy/du & du/dx exist
then
$dy/dx = dy/du \bullet du/dx$

we know that
$d \equiv$ derivative
$\partial \equiv$ partial derivative
so $d \cong \partial$
modifying the above relation
$dy/dx = dy/du \bullet \partial u/\partial x$

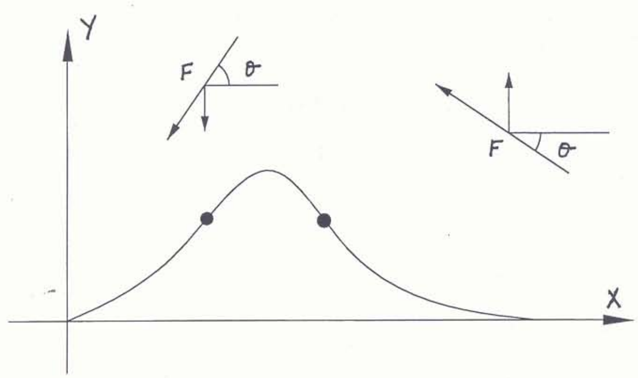
2-8: POWER & INTENSITY

Power: The time rate of doing work.

$$P = \Delta W/\Delta t$$

Intensity: The average power per unit area transmitted across an area A normal to the direction in which the wave is travelling.

$$I = P/A$$



Calculating the rate at which the string transports energy, which is power.
From Fig 2-8, we have for travelling waves

[$x = x_m \cos(\omega t + \phi)$

$Y(x,t) = y_m \sin(kx - \omega t - \phi)$

For zero phase constant ($\phi = 0$)

$\partial y/\partial t = u = -y_m\omega \cos(kx - \omega t)$ .....(2.28)

& $\partial y/\partial x = k\, y_m \cos(kx - \omega t)$ [-ve x direction] .....(2.29)

F_y is vertical component

$F_y = F \sin\theta = F \tan\theta = F\, \partial y/\partial x$ .....(2.30)

We have for vertical case

$P = u F_y$ .....(2.31)

From eqs (2.28) to (2.31), we get

$P = (\partial y/\partial t)(F\, \partial y/\partial x)$

$= F\{-\omega y_m \cos(kx - \omega t)\}\{k y_m \cos(kx - \omega t)\}$

$= F\, y_m^2\, k\omega \cos^2(kx - \omega t)$

since $F = v^2\mu$ & $k = \omega/v$

$\therefore$ **$P = y_m^2\, \mu v \omega^2 \cos^2(kx - \omega t)$** .....(2.32)

now average power, P

$\langle P \rangle = \frac{1}{2} y_m^2\, \mu\omega^2 v$ .....(2.33)

or $\Delta W/\Delta t \propto y_m^2$

$\propto \omega^2$

and not depends upon x or t

$P = \Delta W/\Delta t = F.\Delta d/\Delta t$

$P = F.u = Fu \cos\theta$

for $\theta = 0$

$P = Fu = uF$

$v = \sqrt{F/\mu}$ or $v^2 = F/\mu$

or $F = v^2\mu$

we have

$k = 2\Pi/\lambda$ or $\lambda = 2\Pi/k$

& $v = \lambda\nu = 2\Pi\nu/k = \omega/k$

so $k = \omega/v$

we have

$\sin^2\theta + \cos^2\theta = 1$

or $\frac{1}{2} + \frac{1}{2} = 1$

Calculating Intensity, I

$$I = P/A = y_m^2\, \mu\omega^2/2A \qquad .....(2.34)$$

2-9: DEFINITIONS

Principle of Superposition of Waves: When two waves act upon a body simultaneously they pass each other without disturbing each other, and act upon the particles of the medium quite independent of each other, and their resultant displacement is the resultant of all individual waves.

Complex Waves: When a large number of harmonic waves superpose, the resulting wave is called complex wave.

Fourier Series: The following Series show that any periodic motion of a particle can be represented as a combination of simple harmonic motions.

$$y(x) = A_0 + A_1\sin kx + A_2\sin 2kx + A_3\sin 3kx + \ldots + B_1\cos kx + B_2\cos 2kx + B_3\cos 3kx \qquad ..... (2.35)$$

where y(x) is waveform at a particular time having wavelength λ

& k is wave number equal to $2\pi/\lambda$,

coefficients A and B have definite values for particular periodic motion

Interference: The phenomenon in which the two waves support each other at some points and cancel at others.

Constructive Interference: The interference of two waves, so that they reinforce one another.

Destructive Interference: The interference of two waves , so that they cancel one another.

Standing Waves (Stationary Waves): 1) The resultant of two wave trains of the same wavelength, frequency, and amplitude travelling in opposite directions through the same medium.
2) Waves apparently standing still resulting from two similar wave trains travelling in opposite directions.

Node: A point of no disturbance of a standing wave.

Antinode: A point which oscillate with the maximum amplitude in standing waves.

Diffraction: The bending of waves around the edge of an opening or obstacle.

Reflection: The turning back of a wave from the boundary of a medium.

Refraction: The bending of a wave disturbance as it passes obliquely from one medium into another of different density.

Transmission: The passage of the wave from one medium into the other.

Principle of reflection for tranverse waves: On reflection from a fixed end, a transverse wave undergoes a phase change of 180^0 (or crest changes into trough) and at a free end, a transverse wave is reflected without change of phase.

Resonance: The vibratory motion produced in a body by the influence of another body when their time periods are exactly equal.

Qualitative analysis of standing waves:

Consider two waves $y_1(x,t)$ and $y_2(x,t)$ with zero phase constant travelling in opposite directions (see eqs 2.4 & 2.5):

$y_1(x,t) = y_m \sin(kx - \omega t)$ .....(2.36)

$y_2(x,t) = y_m \sin(kx + \omega t)$ .....(2.36)

from superposition principle, the resultant of the above equations is;

$y(x,t) = y_1(x,t) + y_2(x,t)$

$= y_m \sin(kx - \omega t) + y_m \sin(kx + \omega t)$ .....(2.37)

$= y_m \{\sin(kx - \omega t) + \sin(kx + \omega t)\}$ [sinA + sinB =

$= y_m [2\sin ½ \{2kx\} \cos ½ \{-2\omega t\}]$ [2sin ½ (A + B) cos ½ (A – B)

or $y(x,t) = 2\, y_m \sin kx \cos \omega t$ .....(2.38)

where amplitude of the wave is $2y_m \sin kx$

& maximum amplitude is $2\, y_m$

(for max $\sin kx = 1 \Rightarrow kx = \pi/2, 3\pi/2, 5\pi/2, \ldots$)

since $k = 2\pi/\lambda$, so $x = [kx]\, \lambda/2\pi = \lambda/4, 3\lambda/4, \ldots\ldots.$

& minimum value is for $\sin kx = 0$ i.e. $kx = \pi, 2\pi, 3\pi, \ldots$

it corresponds $x = \lambda/2, \lambda, 3\lambda/2, \ldots$

since $k = 2\pi/\lambda$, so positions of antinodal points corresponding to max amplitude and nodal points corresponding to min amplitudes are:

for antinodes: $x = \lambda/4, 3\lambda/4, 5\lambda/4, 7\lambda/4, \ldots$

for nodes: $x = \lambda/2, \lambda, 3\lambda/2, 2\lambda, 5\lambda/2, \ldots$

SOUND & LIGHT

3-1: BEATS

Definition: 1) The condition whereby two sound waves form an outburst of sound followed by an interval of comparative silence.
2) The periodic alternations of sound between maximum and minimum loudness.

We have for travelling waves

$y(x,t) = y_m \sin(kx - \omega t - \phi)$ [$x = x_m \cos(\omega t + \phi)$]

For fixed position with zero phase constant,

$y(t) = y_m \sin(-\omega t) = -y_m \sin \omega t$

selecting positive direction, we have

$$y(t) = y_m \sin \omega t \quad .....(3.1)$$

Considering sound waves of variation in pressure with time,

$$p(t) = p_m \sin \omega t \quad .....(3.2)$$

$$\text{or } \Delta p_1(t) = \Delta p_m \sin \omega_1 t \quad .....(3.3)$$

$$\& \ \Delta p_2(t) = \Delta p_m \sin \omega_2 t \quad .....(3.4)$$

Applying Principle of Superposition of Waves to eqs (3.3) & (3.4), we get

$\Delta p(t) = \Delta p_1(t) + \Delta p_2(t)$ [$\sin A + \sin B =$

$= \Delta p_m\{\sin \omega_1 t + \sin \omega_2 t\}$ [$2 \sin(A + B)/2 \cos(A - B)/2$

$$= \Delta p_m\{2 \sin(\omega_1 + \omega_2)t/2 \cos(\omega_1 - \omega_2)t/2\} \quad .....(3.5)$$

$$\text{put } (\omega_1 + \omega_2)/2 = \overline{\omega} \ \& \ (\omega_1 - \omega_2)/2 = \omega_{amp} \quad .....(3.6)$$

$$\Delta p(t) = 2\Delta p_m \cos \omega_{amp} t \sin \overline{\omega} t \quad .....(3.7)$$

From relations (3.6), we have

$$2\omega_{amp} = \omega_1 - \omega_2 \quad .....(3.8)$$

which is beat frequency,

$$\therefore \omega_{beat} = \omega_1 - \omega_2 \quad .....(3.9)$$ [$\omega = 2\pi\nu$

$$\text{or } \nu_{beat} = \nu_1 - \nu_2 \quad .....(3.10)$$

i.e., **the number of beats per second is the difference of the frequencies of the component waves.**

3-2: DOPPLER EFFECT

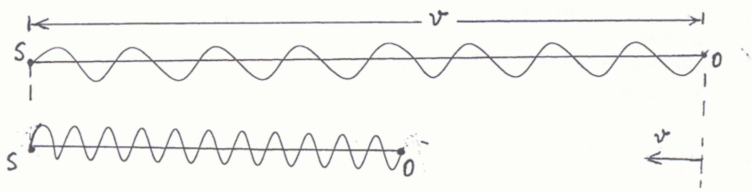
Statement: *The change in the pitch of sound caused by the relative motion of either the source of sound or the listener is called the Doppler effect.*

[**Pitch:** Sensation depending upon frequency by which shrill sound is distinguished from a grave one.]

Explanation: It is observed that the pitch of sound of a whistling train approaching a listener increases and when the train is moving away the pitch decreases.

Qualitative Analysis:

Case 1: Observer is moving towards a stationary Source



When the observer is moving towards the source with velocity, v_o

Here the observer receives more waves in one second than he is at rest.

$S = vt \quad [t = 1/\nu$

or $S = v/\nu$

or $\nu = v/S = v/\lambda$

or $v = \lambda\nu$

or $\lambda = v/\nu$

Additional waves = distance traveled in 1 sec/wavelength

or $\nu_{additional} = v_o/\lambda$

$= v_o/v\,(\nu)$

And the frequency ν' heard by the observer is

$$\nu' = \nu + v_o/v(\nu) = \nu(1 + v_o/v) \qquad \text{.....(3.11)}$$

As $\nu' > \nu$, therefore the pitch of the sound heard by the observer will increase.

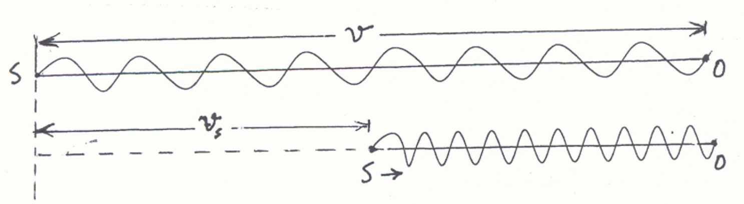
Case 2: Observer is moving away from a stationary Source

If the observer is moving away from the stationary source, the sign of v_o will be reversed, so that

$$\nu' = \nu - v_o/v(\nu) = \nu(1 - v_o/v) \qquad \text{.....(3.12)}$$

As $\nu' < \nu$, therefore the pitch of the sound heard by the observer will decrease.

Case 3: Source is moving towards a stationary Observer



If the source is at rest then

λ = distance which ν waves occupied /number of waves

Number of waves during <u>one second</u> is ν and occupy a length v, so

$\lambda = v/\nu$ [$v = \lambda\nu$

If the source is moving towards the observer, ν waves emitted in the length $(v - v_s)$, so

$$\lambda' = (v - v_s)/\nu \quad .....(3.13)$$

The changed frequency is given by

$$\nu' = v/\lambda' = v\,\nu/(v - v_s)$$

or $$\nu' = \nu(v)/(v - v_s) \quad .....(3.14)$$

As $\nu' > \nu$, therefore the pitch of the sound heard by the observer <u>increases.</u>

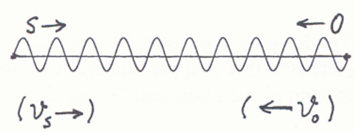
Case 4: <u>Source is moving away from the stationary Observer</u>

If the source is moving away from the observer , the sign of v_s will be reversed with the result that

$$\nu' = \nu(v)/(v + v_s) \quad .....(3.15)$$

As $\nu' < \nu$, therefore the pitch of the sound heard by the observer will <u>decrease</u>.

Case 5: <u>Source & Observer both moving towards each other</u>



When source is moving towards the observer, then from eq (3.13)

$$\lambda' = (v - v_s) / \nu$$

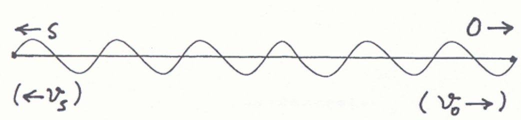
And when the observer is moving, total velocity is, v'

$$v' = v + v_o$$

therefore changed frequency, ν' is

$$\nu' = v'/\lambda' = \nu(v + v_o) /(v - v_s) \quad .....(3.16)$$

Case 6: <u>Source & Observer both moving away from each other</u>



When source is moving away from the observer, changing the sign of v_s

$$\lambda' = (v + v_s) /\nu$$

and when the observer is moving away from the source, changing the sign of v_o

$$v' = v - v_o$$

therefore, changed frequency, ν' is

$$\nu' = v' /\lambda' = \nu(v - v_o) /(v + v_s) \quad .....(3.17)$$

From eqs (3.11) to (3.17) we can form the **General Equation** of Doppler effect,

$$\nu' = \nu (v \pm v_o) / (v \mp v_s) \qquad .....(3.18)$$

Case 0: When source & observer both are stationary

Here $v_o = 0$ & $v_s = 0$, then from eq (3.18), we get

$$\nu' = \nu \qquad .....(3.19)$$

Applications:

1. Applied to light: The frequency of light from certain stars is found to be slightly more and from other stars slightly less than the frequency of the same light emitted from the source on earth. Their velocities can be obtained from this frequency difference.

2. Ultrasonic waves from a bat: A bat determines the location and nature of objects by sending ultrasonic waves.

3. Reflection of radar waves: The frequency of the reflected radar waves is decreased if the plane is moving away and increased if it is approaching. From the observed frequency difference the speed and direction of the plane can be calculated.

4. Detection of submarines: When under-water sound waves are reflected from a moving submarine, we can detect its location.

5. Velocities of earth satellites: These velocities are determined from the Doppler shift in the frequency of their transmitted waves.

Definition:
Doppler Shift: A displacement of lines in the spectra of certain celestial objects towards longer wavelengths (i.e. towards the red end of the visible spectrum). The spectral lines appear at slightly longer wavelengths than they would under normal conditions. It is called red shift. Some objects show a blue shift, indicating movement towards observer. These are due to Doppler effect.

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